ANALITICAL AND NUMERICAL DETERMINATION OF THE EXACT LOCATION OF THE OPTIMUM TORSION AXIS

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ABSTRACT

The paper addresses the issue of the evaluation of a reference axis which in multi-storey buildings can play the same role as the elastic centre in one-storey schemes. In this regard the Authors propose an improvement to the procedure previously suggested by Makarios and Anastassiadis for the evaluation of the position of the *optimum torsion axis*. The above-mentioned researchers based their evaluation on a parametric analysis of asymmetric frame-wall systems and thus defined the location of the optimum torsion axis only approximately. The Authors, instead, face the problem from an analytical point of view and propose mathematical expressions that rigorously define the position of the same axis. Some examples are finally reported aiming at comparing the two approaches and highlighting the improvements brought by the rigorous approach to the approximate determination of the location of the optimum torsion axis.

INTRODUCTION

In order to estimate the effect of the lateral-torsional coupling on the seismic response of in-plan irregular systems, building codes [3], [10], [13] generally propose two different design approaches, based on static and modal analyses respectively. The use of the first approach is, however, only allowed with reference to a restricted category of buildings, which, even if differently described by codes, are commonly characterised by mass and stiffness distributions quite uniform in elevation. Such a limitation is due to the difficulty in extending to generic multi-storey systems the static design procedures widely and thoroughly studied in the past for one-storey systems. According to the above-mentioned methodology the correction of the standard static analysis is performed by means of the introduction of fictitious storey torsional moments [1], [2], [12] basically defined as a function of the structural eccentricity, i.e. of the distance from the centre of mass to the centre of rigidity. The determination of this last point does not generate any problem in one-storey systems. Indeed, as remarked by several researchers [5], [11], in such structures the centre of rigidity, rigorously defined as the point of the floor through which a static force (of arbitrary magnitude and direction) must be applied to cause the deck to translate without torsion [5], always exists. The same location, furthermore, also individuates the point (named *shear centre*) through which the resultant of the storey shear forces passes when the floor is subjected to translation. In addition, the position of the centre of rigidity also coincides with that of the point (named *centre of twist*) that remains stationary in plan when the structure is subjected to torque loading. As a result, with reference to one-storey systems the terms “centre of rigidity”, “centre of twist” and “shear centre” may be used interchangeably and identify a single point in plan, which is called *elastic centre* [6]. Finally, the position of such a point is load independent, since there is a single load resultant acting in one-storey structures.

On the contrary of what has been stated with reference to one-storey systems, the determination of the centre of rigidity may cause some serious problems when dealing with multi-storey buildings. In this regard, let us consider a generic multi-storey system endowed with rigid floors, sustained by vertical resisting elements. Considering as degrees of freedom of the i^{th} deck the floor rotation θ_z and the horizontal displacements u_x and u_y of a point on the floor along two perpendicular directions, the mathematical expression that establishes the existence of the centres of rigidity (and, in the affirmative, defines their coordinates) leads to the following relations [5]:

$$\mathbf{x}_\mathrm{R} = \left(\mathbf{K}_\mathrm{yy} - \mathbf{K}_\mathrm{yx}\mathbf{K}_\mathrm{xx}^{-1}\mathbf{K}_\mathrm{xy}\right)^{-1}\left(\mathbf{K}_\mathrm{y\theta} - \mathbf{K}_\mathrm{yx}\mathbf{K}_\mathrm{xx}^{-1}\mathbf{K}_\mathrm{x\theta}\right) \tag{1a}$$

$$\mathbf{y}_\mathrm{R} = -\left(\mathbf{K}_\mathrm{xx} - \mathbf{K}_\mathrm{xy}\mathbf{K}_\mathrm{yy}^{-1}\mathbf{K}_\mathrm{yx}\right)^{-1}\left(\mathbf{K}_\mathrm{x\theta} - \mathbf{K}_\mathrm{xy}\mathbf{K}_\mathrm{yy}^{-1}\mathbf{K}_\mathrm{y\theta}\right) \tag{1b}$$

where $\mathbf{x}_\mathrm{R}$ and $\mathbf{y}_\mathrm{R}$ are square matrices containing the unknown coordinates of the centre of rigidity and $\mathbf{K}_\mathrm{ij}$ sub-matrices of the building stiffness matrix:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{xx} & \mathbf{K}_{xy} & \mathbf{K}_{x\theta} \\ \mathbf{K}_{yx} & \mathbf{K}_{yy} & \mathbf{K}_{y\theta} \\ \mathbf{K}_{\theta x} & \mathbf{K}_{\theta y} & \mathbf{K}_{\theta\theta} \end{bmatrix} \tag{2}$$

Unfortunately, although the inverse matrix of Eqs.(1a-b) can always be calculated, $\mathbf{x}_\mathrm{R}$ and $\mathbf{y}_\mathrm{R}$ are not always diagonal. If such last matrices are not diagonal, and this is a very common case in practice, the centres of rigidity, as previously rigorously defined, do not exist. As a consequence, also the elastic centre, which must satisfy the properties of the centre of rigidity, of twist and those of the shear centre, does not generally exist in multi-storey buildings.

In such systems points similar to the centre of rigidity may be still identified by means of a slight modification of the above-mentioned corresponding definition, i.e. as points of the decks at which the application of a given distribution of horizontal forces does not cause rotation of the floors. As also reported by Tso [11], in order to avoid confusion with the rigorous centres of rigidity, such points are referred to later as the *generalised centres of rigidity*. As clearly reported by Hejal and Chopra [5] their coordinates may be evaluated by means of the expressions:

$$\underline{\mathbf{x}}_R = \hat{\mathbf{F}}_y^{-1}\,\mathbf{x}_\mathrm{R}^\mathrm{T}\;\;\mathbf{F}_y \tag{3a}$$

$$\underline{\mathbf{y}}_R = \hat{\mathbf{F}}_x^{-1}\,\mathbf{y}_\mathrm{R}^\mathrm{T}\;\;\mathbf{F}_x \tag{3b}$$

being $\hat{\mathbf{F}}_x^{-1}$ and $\hat{\mathbf{F}}_y^{-1}$ the diagonal matrices of the lateral load vectors $\mathbf{F}_\mathrm{x}$ and $\mathbf{F}_\mathrm{y}$ and $\underline{\mathbf{x}}_R$ and $\underline{\mathbf{y}}_R$ the vector forms of the diagonal matrices $\mathbf{x}_\mathrm{R}$ and $\mathbf{y}_\mathrm{R}$ of Eqs.(1a-b).

The development of the products in Eqs.(3a-b) leads to the identification of an important characteristic of the generalised centres of rigidity. Indeed, as evident from the following relations [8], the coordinates of the generalised centre of rigidity of the i^{th} storey (being N the total number of storeys) may be considered as the sum of two contributions:

$$x_{Ri} = x_{Rii} + \sum_{\substack{j=1\\ j\neq i}}^{N} x_{Rij}\frac{F_{yj}}{F_{yi}} \tag{4a}$$

$$y_{Ri} = y_{Rii} + \sum_{\substack{j=1\\ j\neq i}}^{N} y_{Rij}\frac{F_{xj}}{F_{xi}} \tag{4b}$$

The first quota is constituted by the diagonal term of the matrices $\mathbf{x}_\mathrm{R}$ and $\mathbf{y}_\mathrm{R}$ and is independent of the distribution of the horizontal forces. The second, instead, contains the non-diagonal terms of the same matrices multiplied by the ratios of the external forces and is therefore responsible for the dependency of the position of the generalised centres of rigidity on the distribution in elevation of the horizontal loads. Obviously, as a general rule, the more relevant the weight of the diagonal terms of $\mathbf{x}_\mathrm{R}$ and $\mathbf{y}_\mathrm{R}$ with respect to the non-diagonal ones the closer the coordinates of the generalised centres of rigidity to the values of the diagonal terms (note that these elements would represent the coordinates of the rigorous centres of rigidity if all the non-diagonal terms were equal to zero). Nevertheless, if remarkably non-uniform distributions of forces are considered, the contribution of the non-diagonal terms of $\mathbf{x}_\mathrm{R}$ and $\mathbf{y}_\mathrm{R}$ to the determination of the generalised centres of rigidity may be greatly amplified by very high values of the external load ratios. In this case, even when diagonal elements prevail on the others the coordinates of the generalised centres of rigidity may be very different from the diagonal terms of the matrices $\mathbf{x}_\mathrm{R}$ and $\mathbf{y}_\mathrm{R}$.

In the past, analytical studies [5] have shown that, in general, also the shear centre as well as the centre of twist do not exist in multi-storey buildings, i.e. they cannot be evaluated, according to their rigorous definitions, with reference to any set of static forces or torsional couples, respectively. Nevertheless, also in these cases, analogous points may be determined in relation to a specified distribution of external lateral loads, so defining the generalised shear centre and centre of twist. However, it must be observed that, even if referred to a particular set of horizontal forces, generalised centres of rigidity and twist and the generalised shear centres do not usually coincide. In particular, the generalised centres of rigidity seem to be much more sensible than the other generalised reference points to the external load distributions.

In spite of this, several researchers [5], [11] identified a special class of systems (buildings having vertical resisting elements with proportional stiffness matrices) in which the generalised centres of rigidity and twist and the generalised shear centre are lined up along a unique vertical axis and are independent of the lateral load distribution. Such structures are characterised by generalised reference points coinciding with the rigorous ones and thus by the existence of the elastic axis. In addition, if masses are defined by centres aligned along a vertical axis and by radius of gyration equal at all the floors, such buildings are called *regularly asymmetric systems* [5], [9] and are characterised by a single value of the structural eccentricity and of the uncoupled torsional to lateral frequency ratio. In such systems the effect of the lateral-torsional coupling on the elastic seismic response may be evaluated by means of an idealised one-storey system, having the same structural eccentricity and uncoupled torsional to lateral frequency ratio as the multi-storey building. It comes out from such a consideration that, strictly speaking, the static design procedures, developed with the aid of the idealised one-storey model and aiming at correcting its elastic seismic response, would have to be applied to such a category of multi-storey systems only.

Unfortunately, real asymmetric structures rarely fulfil the strict conditions that characterise the above-mentioned buildings. Although many multi-storey buildings present only a slight non-proportionality between the stiffness matrices of the vertical elements, a rigorous identification of the elastic axis is impossible. The same consideration applies to any valid reference axis. Furthermore, the position of the generalised reference points often varies so remarkably from one storey to the other [4], [6]that a rational determination of an axis of the generalised reference centres, representative of the plan-asymmetry of the whole building, is very difficult. This implies that in real, common buildings, even when mass distributions fulfil the properties of regularly asymmetric systems, the equivalent static method proposed by codes generally cannot be used if procedures devoted to the definition of the reference axis location are not clearly stated.

An interesting solution to the problem has in the last years been proposed by Makarios and Anastassiadis [6] who suggested using, as a reference for the calculation of the structural eccentricity, the *optimum torsion axis*, defined as the vertical line that joins the points of the floors where the equivalent seismic forces must be applied in order to minimise the sum of the squares of the deck rotations. In their own proposition the above-mentioned researchers defined the location of such an axis by means of an approximate method, the validity of which is based on a parametric analysis of asymmetric frame-wall systems. This paper now faces the same problem from an analytical point of view with the aim of proposing the mathematical expressions which, with precision and scientific rigour, define the position of the optimum torsion axis. Some examples are furthermore reported to compare the two approaches and to identify the improvements brought by the rigorous approach to the approximate determination of the location of the optimum torsion axis.

THE OPTIMUM TORSION AXIS

On the basis of observations similar to those briefly reported in the previous section, some years ago Anastassiadis and Makarios strictly denied the possibility of using the generalised centres of rigidity, twist and shear centres as a reference for the definition of the structural eccentricity in multi-storey buildings. With the aim of providing for the lack of reference points equivalent to the elastic centre the same authors generalised the concept of the elastic axis defining the optimum torsion axis [6]. Such an axis always exists and coincides with the elastic axis when the latter exists. Furthermore, on the contrary of what has been stated with reference to the centres of rigidity, its position is only a little influenced by the distribution of the horizontal forces. The concept of the optimum torsion axis quite naturally results from the examination of the response of a multi-storey system undergoing the action of a given distribution of horizontal forces disposed along a vertical plane. Indeed, if the elastic axis exists the determination of the position of the loading plane able to nullify the deck rotations is always possible. Instead, if the elastic axis does not exist, a position of the loading plane may be found which minimises the deck rotations of the building. This occurs when the following expression is satisfied:

$$\Theta = \sum_{i=1}^{N} \theta_i^2 = \text{minimum} \tag{5}$$

Such a relationship constitutes the mathematical representation of the optimum torsion criterion, which states that the torsion of a building is optimal when the sum of squares of the deck rotations is minimal. Owing to the nature of the function to be minimised, the minimum is always greater or equal to zero: in particular, the equality to zero is satisfied only in the presence of asymmetric buildings having an elastic axis.

<u>Approximated evaluation</u>

In their own studies [6], [7] do not face the problem of the rigorous mathematical treatment of Eq.(5), considered by the same researchers as “practically impossible due to the great variety and complexity of multi-storey building structures”. Alternatively, with the aim of proposing an approximate method for the calculation of the position of the optimum torsion axis, they carry out a parametric analysis of frame-wall systems. In particular, they analyse systems (referred to as regular) in which the geometrical and mechanical characteristics of the substructures remain constants or present a smooth variation along the height of the model. From this investigation Anastassiadis and Makarios highlight that when the minimum of Θ is attained the deck rotation varies along the height from positive to negative values, being null at a level z_o contained in a narrow range of values (from about 0.75 to 0.85 times the height H of the building). This observation leads the same researchers to the simplified formulation of the optimum torsion criterion, according to which the torsion of the building is optimal when the nullification of the deck rotations takes place at the level $z_o = 0.80\ H$, approximately.

On the basis of the above-mentioned formulation, an important property of multi-storey buildings allows the identification of the approximated position of the optimum torsion axis by means of an easy analytical procedure. With the aim of demonstrating such a statement with reference to a generic multi-storey building, let two sets of external loads applied to the decks of the same system be considered: the first **F** constituted by horizontal forces $F_1, F_2, \cdots, F_N$ belonging to a vertical plane and the second **M** constituted by torsional couples such that $M_1 = 1 \times F_1$, $M_2 = 1 \times F_2, \cdots$, $M_N = 1 \times F_N$. Owing to a reciprocity proposition (extensively reported by Anastassiadis and Makarios [6]), at any floor of the above-mentioned generic system the deck rotation $\theta_{i,F}$ caused by the forces **F** is numerically equal to the component of the displacement $u_{i,M}$, due to the couples **M**, along the trace of the loading plane on the deck. Hence, it immediately results that if a point of the deck does not undergo any displacement when the building is subjected to the couples **M** (i.e. the point under consideration is a twist centre) the deck does not undergo any rotation when the building is subjected to the forces **F** applied to any vertical plane passing through this point. Therefore, the optimum torsion axis, as defined in the simplified formulation, coincides with the vertical axis passing through the centre of twist of the deck characterised by a level equal to 0.80 *H*, approximately.

Furthermore, rotating the plane of the forces **F** around this axis the deck rotation at level 0.80 H is always zero while Θ shows a little oscillation around the value relative to the direction first considered. As a direct consequence of the reciprocity proposition, being defined as P_i the intersections of the optimum torsion axis on the decks, also the mean of the sum of squares of the displacements of the points P_i produced by the application of the couples **M** highlights some variation, as a function of the direction of the plane on which the displacements are projected.

EXACT EVALUATION OF THE OPTIMUM TORSION AXIS

While Anastassiadis and Makarios propose to determine the position of the optimum torsion axis by using the above-mentioned simplified criterion, the validity of which is sustained by the results of a numerical parametric analysis, the Authors demonstrate that the exact position of the same axis can be obtained by means of two analytical approaches characterised by different good qualities. The first, which provides analytical formulas based on the manipulation of matrices that define the stiffness properties of the structure, will be called the *analytical approach*. The second, which involves the evaluation of the static response of the building to the seismic forces, will be called the *numerical approach*. Both methods are easily applicable to generic asymmetric multi-storey buildings endowed with rigid floors sustained by resisting elements (frames, walls, etc.), also when arranged along non-orthogonal directions.

Analytical approach

Given an orthogonal reference system *0XYZ*, having *X* and *Y* axes parallel to the principal axes of the resisting elements, let the degrees of freedom of the system be defined by the deck rotations $\boldsymbol{\theta}_z$ and by the horizontal displacements $\mathbf{u}_x$ ed $\mathbf{u}_y$ along the coordinate axes of the intersections of the vertical axis on the decks. Let $\mathbf{F}_y$ represent a set of horizontal forces belonging to a vertical plane parallel to the *Y*-axis and intersecting the *X*-axis at a position *x*. The effect of $\mathbf{F}_y$ on the structure is equal to the sum of two quotas: the first determined by the forces

$\mathbf{F}_y$ applied at the intersections of the vertical Z-axis on the decks and the second determined by a distribution of torsional moments acting on the floors and having intensity equal to $x\,\mathbf{F}_y$. The equations of equilibrium of the system are:

$$\mathbf{K}_{xx}\mathbf{u}_x + \mathbf{K}_{xy}\mathbf{u}_y + \mathbf{K}_{x\theta}\,\theta_z = \mathbf{0} \tag{6a}$$

$$\mathbf{K}_{yx}\mathbf{u}_x + \mathbf{K}_{yy}\mathbf{u}_y + \mathbf{K}_{y\theta}\,\theta_z = \mathbf{F}_y \tag{6b}$$

$$\mathbf{K}_{\theta x}\mathbf{u}_x + \mathbf{K}_{\theta y}\mathbf{u}_y + \mathbf{K}_{\theta\theta}\,\theta_z = x\,\mathbf{F}_y \tag{6c}$$

where $\mathbf{K}_{ij}$ represents the different sub-matrices of the building stiffness matrix $\mathbf{K}$.

Obtaining $\mathbf{u}_x$ from Eq.(6a) and substituting into Eqs.(6b) and (6c), the previous equations of equilibrium can be rewritten as follows:

$$\mathbf{u}_x = -\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\,\mathbf{u}_y - \mathbf{K}_{xx}^{-1}\mathbf{K}_{x\theta}\,\theta_z \tag{7a}$$

$$\mathbf{u}_y = \left(\mathbf{K}_{yy} - \mathbf{K}_{yx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)^{-1}\mathbf{F}_y - \left(\mathbf{K}_{yy} - \mathbf{K}_{yx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)^{-1}\left(\mathbf{K}_{y\theta} - \mathbf{K}_{yx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{x\theta}\right)\theta_z \tag{7b}$$

$$\left(\mathbf{K}_{\theta y} - \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)\mathbf{u}_y + \left(\mathbf{K}_{\theta\theta} - \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{x\theta}\right)\theta_z = x\,\mathbf{F}_y \tag{7c}$$

Hence, remembering the particular form of the $\mathbf{x}_R$ matrix shown by Eq.(1a), we can transform Eq.(7b) in:

$$\mathbf{u}_y = \left(\mathbf{K}_{yy} - \mathbf{K}_{yx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)^{-1}\mathbf{F}_y - \mathbf{x}_R\theta_z \tag{8}$$

and Eq.(7c) in:

$$\begin{aligned}\left(\mathbf{K}_{\theta y} - \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)\left(\mathbf{K}_{yy} - \mathbf{K}_{yx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)^{-1}\mathbf{F}_y - \left(\mathbf{K}_{\theta y} - \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)\mathbf{x}_R\theta_z +\\ + \left(\mathbf{K}_{\theta\theta} - \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{x\theta}\right)\theta_z = x\,\mathbf{F}_y\end{aligned} \tag{9}$$

Furthermore, owing to the symmetry of the building stiffness matrix $\mathbf{K}$, the $\mathbf{K}_{ij}$ matrices fulfil the conditions:

$$\begin{aligned}&\mathbf{K}_{xx} = \mathbf{K}_{xx}^T \qquad &\mathbf{K}_{yy} = \mathbf{K}_{yy}^T \qquad &\mathbf{K}_{\theta\theta} = \mathbf{K}_{\theta\theta}^T\\ &\mathbf{K}_{xy} = \mathbf{K}_{yx}^T \qquad &\mathbf{K}_{x\theta} = \mathbf{K}_{\theta x}^T \qquad &\mathbf{K}_{y\theta} = \mathbf{K}_{\theta y}^T\end{aligned} \tag{10}$$

and thus, from simple mathematical operations:

$$\begin{aligned}&\left\{\left(\mathbf{K}_{\theta y} - \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)\left(\mathbf{K}_{yy} - \mathbf{K}_{yx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)^{-1}\right\}^T =\\ &= \left(\mathbf{K}_{yy} - \mathbf{K}_{yx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\right)^{-1}\left(\mathbf{K}_{y\theta} - \mathbf{K}_{yx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{x\theta}\right) = \mathbf{x}_R^T\end{aligned} \tag{11}$$

Substituting Eq.(11) in Eq.(9) leads to:

$$\mathbf{x}_R^T\ \mathbf{F}_y + \left(\mathbf{K}_{\theta\theta} - \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{x\theta} - \mathbf{K}_{\theta y}\mathbf{x}_R + \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\mathbf{x}_R\right)\theta_z = x\ \mathbf{F}_y \tag{12}$$

and, hence, to the deck rotations:

$$\theta_z = \mathbf{A}_1^{-1}\left(x\ \mathbf{I} - \mathbf{x}_R^T\right)\mathbf{F}_y \tag{13}$$

being **I** the unit matrix and:

$$\mathbf{A}_1 = \mathbf{K}_{\theta\theta} - \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{x\theta} - \mathbf{K}_{\theta y}\mathbf{x}_R + \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xy}\mathbf{x}_R \tag{14}$$

The inverse matrix $\mathbf{A}_1^{-1}$ must always exist because Eq.(12) admits a unique solution in terms of deck rotations θ_z. Furthermore, being symmetric matrix A_1 also the inverse matrix $\mathbf{A}_1^{-1}$ is symmetric.

The square of the deck rotations is therefore:

$$\begin{aligned}\Theta = \theta_z^T\theta_z &= \mathbf{F}_y^T\left(x\ \mathbf{I} - \mathbf{x}_R\right)\mathbf{A}_1^{-1}\mathbf{A}_1^{-1}\left(x\ \mathbf{I} - \mathbf{x}_R^T\right)\mathbf{F}_y^T = \\ &= \mathbf{F}_y^T\mathbf{B}_1\mathbf{F}_y x^2 - \mathbf{F}_y^T\left(\mathbf{B}_1\ \mathbf{x}_R^T + \mathbf{x}_R\mathbf{B}_1\right)\mathbf{F}_y x + \mathbf{F}_y^T\mathbf{x}_R\mathbf{B}_1\mathbf{x}_R^T\mathbf{F}_y\end{aligned} \tag{15}$$

where $\mathbf{B}_1 = \mathbf{A}_1^{-1}\mathbf{A}_1^{-1}$.

But, owing to the equality:

$$\mathbf{F}_y^T\mathbf{x}_R\mathbf{B}_1\ \mathbf{F}_y = \mathbf{F}_y^T\mathbf{B}_1\ \mathbf{x}_R^T\mathbf{F}_y \tag{16}$$

Eq.(15) may be written as:

$$\Theta = \mathbf{F}_y^T\mathbf{B}_1\ \mathbf{F}_y x^2 - 2\ \mathbf{F}_y^T\mathbf{B}_1\ \mathbf{x}_R^T\mathbf{F}_y x + \mathbf{F}_y^T\mathbf{x}_R\mathbf{B}_1\ \mathbf{x}_R^T\mathbf{F}_y \tag{17}$$

Eq.(17) shows that the sum of squares of the deck rotations depends on x by means of a parabolic law. Furthermore, because the function Θ assumes positive values only and the parabola turns the convexity to the X-axis (Appendix A) the minimum of Θ is obtained in correspondence of the abscissa:

$$x_o = \frac{\mathbf{F}_y^T\mathbf{B}_1\ \mathbf{x}_R^T\mathbf{F}_y}{\mathbf{F}_y^T\mathbf{B}_1\ \mathbf{F}_y} \tag{18}$$

Analogously to what was previously described we can obtain, with reference to horizontal forces acting along the X-direction, the coordinate of the intersection of the loading plane on the Y-axis which minimises the sum of squares of the deck rotations:

$$y_o = \frac{\mathbf{F}_x^T\mathbf{B}_2\ \mathbf{y}_R^T\mathbf{F}_x}{\mathbf{F}_x^T\mathbf{B}_2\ \mathbf{F}_x} \tag{19}$$

where $\mathbf{B}_2 = \mathbf{A}_2^{-1}\mathbf{A}_2^{-1}$ and $\mathbf{A}_2 = \mathbf{K}_{\theta\theta} - \mathbf{K}_{\theta y}\mathbf{K}_{yy}^{-1}\mathbf{K}_{y\theta} + \mathbf{K}_{\theta x}\mathbf{y}_R + \mathbf{K}_{\theta y}\mathbf{K}_{yy}^{-1}\mathbf{K}_{yx}\mathbf{y}_R$.

It can be also easily demonstrated that when the resisting elements are arranged along the two orthogonal axes X and Y, owing to the property $\mathbf{K}_{xy} = \mathbf{0}$ the $\mathbf{x}_R$ and $\mathbf{y}_R$ matrices assume the following form:

$$\mathbf{x}_R = \mathbf{K}_{yy}^{-1}\mathbf{K}_{y\theta} \tag{20a}$$

$$\mathbf{y}_R = -\mathbf{K}_{xx}^{-1}\mathbf{K}_{x\theta} \tag{20b}$$

Hence we obtain:

$$\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{K}_{\theta\theta} - \mathbf{K}_{\theta x}\mathbf{K}_{xx}^{-1}\mathbf{K}_{x\theta} - \mathbf{K}_{\theta y}\mathbf{K}_{yy}^{-1}\mathbf{K}_{y\theta} \tag{21a}$$

$$\mathbf{B}_1 = \mathbf{B}_2 \tag{21b}$$

On the basis of the described demonstration it is highlighted that the determination of the exact position of the optimum torsion axis requires only the knowledge of the external reference forces and that of the building stiffness matrix. The formulas obtained may be easily implemented on personal computers within structural programs in which the above-mentioned quantities are anyway necessary and therefore known.

Numerical approach

The procedure shown in the previous section may be awkward to use if it is not implemented within structural programs. For this reason a more simple procedure for the exact determination of the location of the optimum torsion axis has been developed. This more practical approach starts from the observation that, if the deck rotations produced by the lateral forces $\mathbf{F}_y$ applied at the origin of the reference system and those caused by torsional couples $\mathbf{M} = 1 \cdot \mathbf{F}_y$ are known, the evaluation of the building stiffness matrix is not explicitly required in order to determine the relation $\Theta(x)$. In fact, named θ_{Fy} and θ_M the vectors containing the two aforementioned sets of rotations respectively, the rotations of the decks produced by the horizontal forces $\mathbf{F}_y$ acting at a position x may be written as:

$$\theta_z(x) = \theta_{F_y} + x\theta_M \tag{22}$$

Hence, by multiplying the vector $\theta_z(x)$ by itself we immediately obtain that the sum of squares of the decks rotations $\Theta(x)$ is a quadratic function of the loading plane position having general expression:

$$\Theta = \mathrm{a}\,x^2 + \mathrm{b}\,x + \mathrm{c} \tag{23}$$

where the a, b and c coefficients depend on the elastic response of the structure through the following relationships:

$$\mathrm{a} = \theta_M^T\theta_M \tag{24a}$$

$$\mathrm{b} = 2\ \theta_{F_y}^T\theta_M \tag{24b}$$

$$c = \theta_{F_y}^T \theta_{F_y} \tag{24c}$$

Owing to this, the coordinates of the optimum torsion axis corresponding to the vertex of the parabola are:

$$x_o = -\frac{b}{2a} = -\frac{\theta_{F_y}^T \theta_M}{\theta_M^2} \tag{25a}$$

$$y_o = \frac{\theta_{F_x}^T \theta_M}{\theta_M^2} \tag{25b}$$

As evident from the previous demonstration, the present method requires only the application of three static analyses in order to determine the rotations θ_{Fx}, θ_{Fy}, θ_M. Furthermore, it is to be noted that the structural response to any set of horizontal forces may be obtained by linearly combining the results of two of the three static analyses previously performed to determine the optimum torsion axis. Therefore, it is not necessary to carry out any further static analysis to determine the structural response to the design seismic forces, applied according to the design eccentricities suggested by codes.

APPLICATIONS

The location of the optimum torsion axis has been calculated by means of both the approximated and exact formulations with reference to different types of non-regularly asymmetric systems. The typology of buildings first considered (type 1) is endowed with frame-wall structures (Fig. 1a). For the sake of simplicity the plan of the systems under consideration, rectangular (28.50 × 12.50 m), is supposed to be equal at all floors and the mass hypothesised uniformly distributed in plan. The buildings are ten-storeys high and characterised by an interstorey height equal to 3.20 m at all storeys. Frames are shear-type and symmetrically disposed about the coordinate axes; walls, instead, are constituted by two couples of walls (one

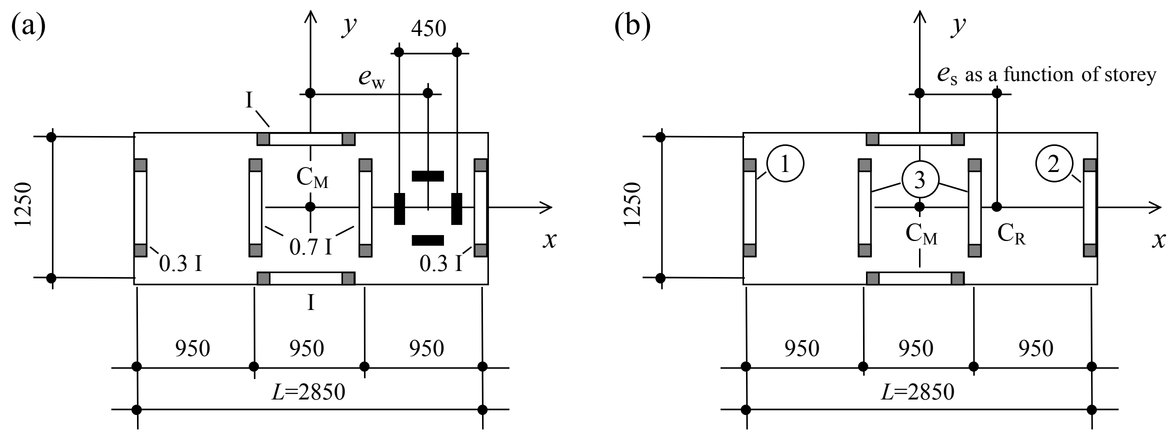

Figure 1. Plan of non-regularly asymmetric buildings.
(a) frame-wall system (type 1), (b) framed system (type 2)

for each direction) characterised by a radius of gyration of stiffness constant at all storeys and equal to 2.25 m. The rigidity of the frames, distributed in plan as shown in Figure 1a, is one half of the value that, in the corresponding torsionally balanced frame structure, determines a first translational period of vibration of 1 s. On the basis of such a result, the dimensions of the walls have been calibrated so that the torsionally balanced coupled wall-frame system has the first translational period of vibration equal to 1 s. With the aim of covering small, moderate and quite high values of structural eccentricity of the whole system (frames plus walls), the distance from the centre of rigidity of the walls to the centre of mass has been gradually increased at all storeys from 0 to 10 m (0-35 % of the floor dimension along the X-axis).

The second typology (type 2) of non-regularly asymmetric buildings has instead a structure constituted by shear-type frames only (Fig. 1b). Both the geometry of the buildings and the characteristics of the mass are equal to those already described for the first typology of systems. But, differently from the previous schemes, in these all frames have the same lateral stiffness. The dimensions of the resisting elements and the geometrical distribution of the frames are such that, at this stage of the design process, the systems (which are still symmetric) are characterised by a first translational period of vibration of 1 s and by a torsional to lateral frequency ratio equal to 1. In such configurations the asymmetry has been introduced by reducing the stiffness of some frames (to one sixth of the original value) at some storeys only. In particular (Fig. 2), in a first set of systems the reduction of rigidity is produced in the element number two, in correspondence to one storey only; such systems are defined as sub-type A_i, being *i* the storey where the reduction in stiffness takes place. A second system has been obtained by reducing the stiffness of the element number two at the eighth and ninth storeys contemporarily (sub-type B). Finally, in a third system the reduction of stiffness has been applied to the element number one at the eighth storey and to the element number two at the ninth storey (sub-type C). These last two schemes have been conceived so as to worsen the prediction of the optimum torsion axis obtained by means of the approximated formulation, the reliability of which is strictly related to the approximation of the centre of twist at 80% of

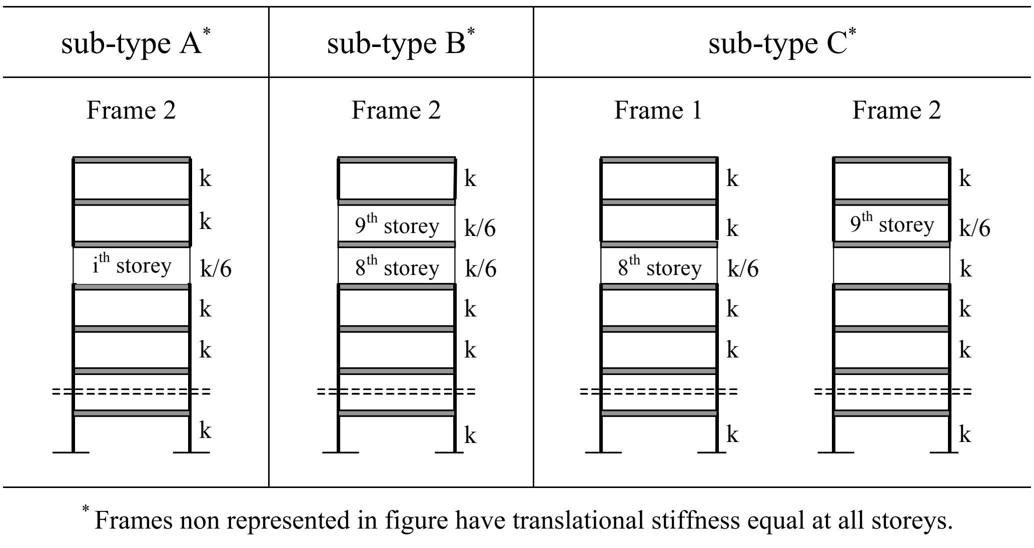

Figure 2. Systems type 2: irregularities in the distribution of lateral stiffness (k) between frames.

the total height of the building to the real position of the optimum torsion axis.

The results of the analyses performed on the above-mentioned systems suggest some interesting considerations. First of all, the location of the centre of rigidity (shown in Figs. 3 and 4 with reference to a triangular distribution of horizontal forces) highlights remarkable differences along the height of the systems, belonging to either the first or the second typology of buildings. As already affirmed by other authors [6], [8], [12], its sensitivity to even low values of structural eccentricity and its variation with the load distribution (not shown in figure) bring us to think that it is not apt to represent a valid reference point for the application of the

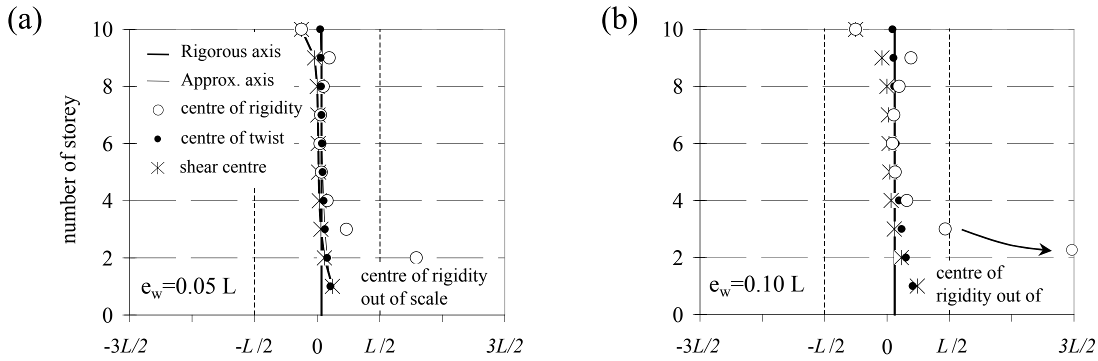

Figure 3. Position of the reference points: frame-wall systems (type 1).

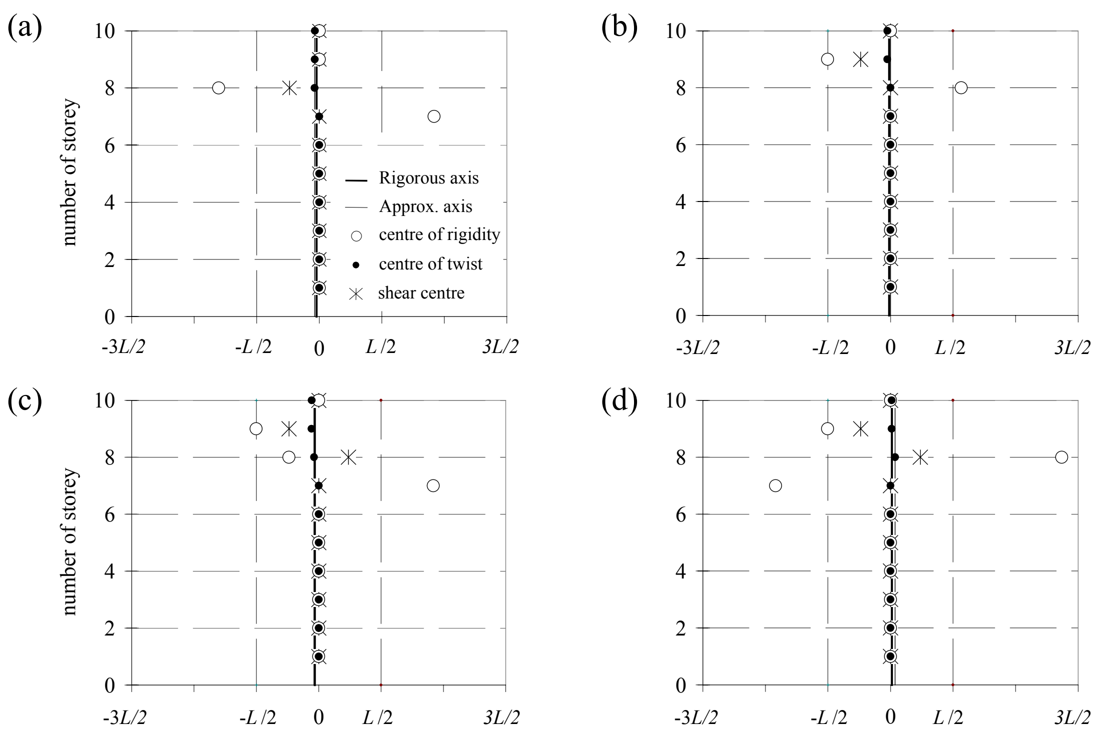

Figure 4. Position of the reference points: framed systems (type 2)
(a) sub-type A_8: reduction at the eighth storey; (b) sub-type A_9: reduction at the ninth storey; (c) sub-type B; (d) sub-type C.

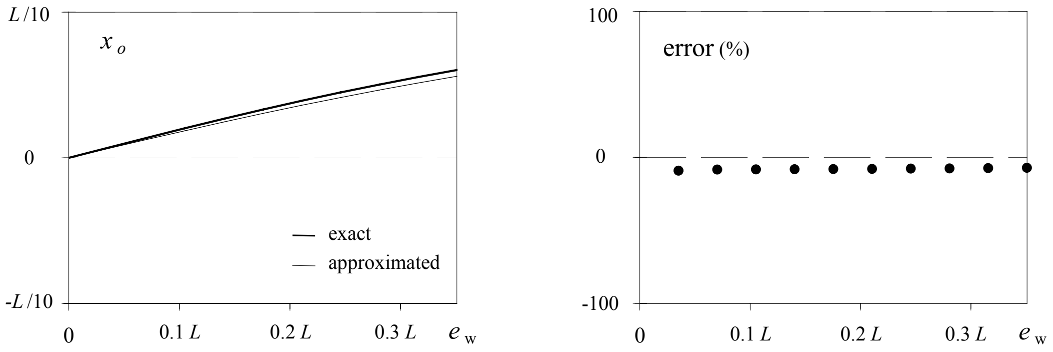

Figure 5. Approximated and exact positions of the optimum torsion axis: systems type 1.

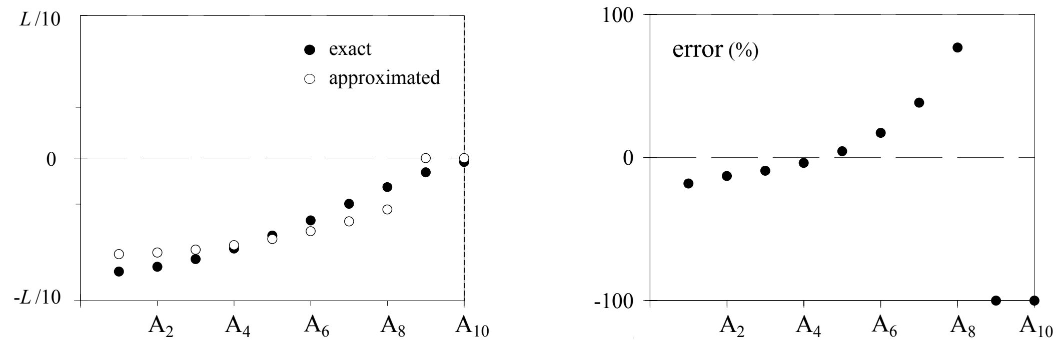

Figure 6. Approximated and exact positions of the optimum torsion axis: systems type 2 (sub-type A).

static analysis to non-regularly asymmetric buildings. At the same time the centre of twist seems to be only a little sensitive to the type of structural irregularity (being continuous or concentrated) or to the lateral load distribution and often quite close to the optimum torsion axis.

The difference between the approximated and exact position of the optimum torsion axis is finally shown in Figures 5 and 6 with reference to the schemes under examination. In the frame-wall systems (Fig. 5) the position of the optimum torsion axis seems to be well evaluated by the approximate procedure with an error close to 10%, whatever the value of the structural eccentricity of the walls. Nevertheless, it should be noted that, although the distance of the centre of rigidity of the walls from the centre of mass is increased up to 35 % of the floor dimension along the X-axis, the results of such analyses correspond to small values of the eccentricity of the optimum torsion axis only (not greater than 6.25% of the floor dimension).

Greater differences between the approximated and exact evaluations of the position of the optimum torsion axis are obtained from the study of buildings belonging to type 2, characterised by concentrated reductions of rigidity. For such systems (sub-type A) Figure 6 shows that the error depends on the storey where the reduction of rigidity has been applied, varying from about 20% to 100%. In particular, the numerical analyses highlight that the approximated formulation predicts null structural eccentricity when the variation of rigidity applies to a storey higher than the eighth, while the correct formulation identifies structural eccentricities that can determine, owing to their values, non-negligible variations in the response. Conversely, quite low errors are noticed if the reduction of rigidity is considered at the lower storeys.

Obviously, also for systems with asymmetry sub-type B and C the numerical analyses highlight relevant errors, equal to 20% and 248% of the exact value respectively (in these cases the approximated structural eccentricity is greater than the correct value). In spite of this, the Authors remark that sometimes, in particular for the last value, relevant errors occur because of the low values of the exact structural eccentricity. Nevertheless, by means of the results of the presented examples they also want to underline that configurations exist that are not very apt to be well analysed by means of the approximated method and that often the differences in the position of the optimum torsion axis may be so great as to determine serious underestimations in the structural response.

CONCLUSIONS

In this paper the Authors address the problem of the exact evaluation of the position of the optimum torsion axis, which constitutes one of the essential elements for the study and design of non-regularly asymmetric multi-storey buildings. The study proposes itself as a refinement of a previous research carried out by Anastassiadis and Makarios who, on the basis of a parametric analysis of frame-wall systems, have in the past proposed an approximate evaluation of the position of the same axis.

Two exact procedures, characterised by different good qualities, are presented to reach the same goal.

- The first requires the knowledge of the building stiffness matrix and is, for this reason, particularly useful if implemented within a structural program (where such data are already available).
- The second is based on the results of three static analyses and is therefore easily applicable even if structural programs do not allow the previous analytical evaluation.

By means of some examples the Authors demonstrate that great differences between the approximated (proposed by Anastassiadis and Makarios) and exact evaluations of the position of the optimum torsion axis are possible and that, therefore, the application of the proposed procedure is in general desirable. Nevertheless, they also show that in some systems, similar to those considered by Anastassiadis and Makarios in their research, the two methods may give analogous results.

Finally, the Authors want to underline that the examples contained in this paper are not intended to cover all possible cases of non-regularly asymmetric structures. The selected systems only want to constitute a limited basis of less or more realistic structural schemes, of relevant interest for the application of the described approximated and exact procedures.

APPENDIX A

Owing to the symmetry of matrix $\mathbf{A}_1^{-1}$ the eigenvalues of such a matrix must be real. Furthermore, the inverse matrix of $\mathbf{A}_1^{-1}$ always exists and thus such eigenvalues must be also all different from zero. An orthogonal matrix P exists such that:

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}_1^{-1}\mathbf{P} = \hat{\mathbf{A}}_1^{-1} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ldots & & \\ & & & \ldots & \\ & & & & \lambda_n \end{bmatrix}$$

in which $\lambda_1, \lambda_2 .. \lambda_n$ are the eigenvalues of $\mathbf{A}_1^{-1}$.

If we multiply the matrix $\mathbf{A}_1^{-1}$ by itself we obtain:

$$\hat{\mathbf{A}}_1^{-1}\hat{\mathbf{A}}_1^{-1} = \mathbf{P}^{\mathrm{T}}\mathbf{A}_1^{-1}\mathbf{P}\ \ \mathbf{P}^{\mathrm{T}}\mathbf{A}_1^{-1}\mathbf{P} = \mathbf{P}^{\mathrm{T}}\mathbf{A}_1^{-1}\mathbf{A}_1^{-1}\mathbf{P} = \hat{\mathbf{B}}_1 = \begin{bmatrix} \lambda_1^2 & & & & \\ & \lambda_2^2 & & & \\ & & \ldots & & \\ & & & \ldots & \\ & & & & \lambda_n^2 \end{bmatrix}$$

Therefore, the eigenvalues of the matrix $\mathbf{B}_1$ are all real and positive and thus the quadratic form $\mathbf{F}_y^{\mathrm{T}}\mathbf{B}_1\mathbf{F}_y$ is always positive, whatever the vector of forces $\mathbf{F}_y$ is. Consequently (see Eq. 17), the convexity of the parabola under examination is turned to the X- axis.

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